K25P 2017

Reg. No. : ....

Name : ....

II Semester M.Sc. Degree (C.B.S.S. – Supplementary)
Examination, April 2025
(2021 and 2022 Admissions)
MATHEMATICS
MAT 2C 07 : Measure and Integration

Time : 3 Hours Max. Marks : 80

PART – A

Answer **any four** questions from this Part. **Each** question carries **4** marks. **(4×4=16)**

1. Prove that $m^*([x]) = 0$ for any $x \in R$.
2. Prove that every Borel set is measurable.
3. Prove that if f and g are measurable. $|f| \leq |g|$ a.e., and g is integrable, then f is integrable.
4. Prove that the intersection of hereditary σ-rings is again a σ-ring.
5. Let f = g a.e. (μ), where μ is a complete measure. Show that if f is measurable, so is g.
6. If $\rho(f, g) = \|f - g\|_p$, then prove that, for $p \geq 1$, ρ is a metric on $L^p(\mu)$.

PART – B

Answer **any four** questions from this part without omitting **any Unit**.
Each question carries **16** marks. **(4×16=64)**

Unit – I

7. a) Let M be the class of Lebesgue measurable sets. Prove that M is a σ-algebra.
   b) Prove that Lebesgue outer measure is countably additive on disjoint measurable sets.

8. Prove that the following statements regarding the set E are equivalent
   a) E is measurable
   b) $\forall \varepsilon > 0$, there exists O, an open set, $O \supseteq E$ such that $m^*(O - E) \leq \varepsilon$
   c) there exists G, a G_δ-set, $G \supseteq E$ such that $m^*(G - E) = 0$
   d) $\forall \in > 0$, there exists F, a closed set, $F \subseteq E$ such that $m^*(E - F) \leq \in$
   e) there exists F, a F_σ-set, $F \subseteq E$ such that $m^*(E - F) = 0$.

P.T.O.

9. a) Show that there exists uncountable sets of zero measure.

   b) Prove that there exists a non-measurable set.

Unit – II

10. Define integrable functions. Let f and g be integrable functions. Then prove the following.

    a) af is integrable and $\int afdx = a\int fdx$

    b) f + g is integrable and $\int (f+g)dx = \int fdx + \int gdx$

    c) If f = 0 a.e., then $\int fdx = 0$

    d) If $f \leq g$ a.e., then $\int fdx \leq \int gdx$

    e) If A and B are disjoint measurable sets, then $\int_A fdx + \int_B fdx = \int_{A\cup B} fdx$.

11. a) State and prove Lebesgue's Dominated Convergence Theorem.

    b) Show that $\lim_{\beta\to\infty} \int_a^b f(x)\sin\beta x dx = 0$.

12. a) The outer measure μ^* on H(R) defined by μ on R and the corresponding outer measure defined by $\bar{\mu}$ on S(R) and $\bar{\mu}$ on S^* are the same.

    b) If μ is a σ-finite measure on a ring R, then prove that it has a unique extension to the σ-ring S(R).

Unit – III

13. a) Let $0 < p < 1$ and $f \geq 0$, $g \geq 0$, f, $g \in L^p(\mu)$. Show that $\|f+g\|_p \geq \|f\|_p + \|g\|_p$.

    b) Let $\{f_n\}$ is a sequence in $L^\infty(\mu)$ such that $\|f_n - f_m\|\infty \to 0$ as n, $m \to \infty$, then prove that there exist a function f such that $\lim f_n = f$ a.e., $f \in L^\infty(\mu)$ and $\lim\|f_n - f\|\infty = 0$.

14. Let $p \geq 1$ and f, $g \in L^p(\mu)$, then prove that $\left(\int |f+g|^p d\mu\right)^{\frac{1}{p}} \leq \left(\int |f|^p d\mu\right)^{\frac{1}{p}} + \left(\int |g|^p d\mu\right)^{\frac{1}{p}}$.

    When does the equality occur ? Justify your answer.

15. State and prove Holder's inequality. When does the equality occur ? Justify.
